Q/. Examine the continuity of the function at $x=0$

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solⁿ. LHL at $x=0$ = $f(0^-) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0}\left[(0-h)\sin\left(\frac{1}{0-h}\right)\right]$

$= \lim_{h\to 0}\left[-h\sin\left(\frac{1}{-h}\right)\right] = 0 \times \{\text{value lies from } -1 \text{ to } 1\}$

$= 0$ ——— (i)

RHL at $x=0$ = $f(0^+) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}\left[(0+h)\sin\left(\frac{1}{0+h}\right)\right]$

$= \lim_{h\to 0}\left[h\sin\left(\frac{1}{h}\right)\right] = 0 \times \{\text{value lies from } -1 \text{ to } 1\}$.

$= 0$ ——— (ii)

V.o.f. at $x=0 = f(0) = 0$ ——— (iii)

from (i), (ii) & (iii): $f(x)$ is continous at $x=0$.

Q/ Let a f is defined in the domain R s.t.

$$f(x) = \begin{cases} 1 & ; \text{ if } x \text{ is rational} \\ -1 & ; \text{ if } x \text{ is irrational} \end{cases}$$

then, show that $f(x)$ is discontinous every where.

Solⁿ. Case (i)

Let a, be a rational number

$\Rightarrow f(a) = 1$ ——— (i) given.

we know that if a is rational, then there are infinite number of irrational numbers in the neighbourhood of $x = a$.

$\Rightarrow$ a+h may be irrational

$\Rightarrow f(a+) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} (-1) = -1$ ——— (2)

Similarly a-h may be irrational

$\Rightarrow f(\bar{a}) = -1$ ——— (3)

& $f(a) = 1$ ——— (4)

Similarly

Case(ii) let a is irrational.

$\Rightarrow$ (a-h) may be rational

$\Rightarrow \lim_{h\to 0} f(a-h) = \lim_{h\to 0} (+1) = +1$ ——— (5)

& $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} (+1) = +1$ ——— (6)

And $f(a) = -1$ ——— (7)

$\Rightarrow$ In both the cases LHL = RHL $\neq$ V.O.F. for rational and irrational numbers, both

$\Rightarrow$ f(x) is discontinuous for rationals and irrationals both..

Q/ If $f(x) = \lim_{n\to\infty}\left[\dfrac{x^n}{1+x^n e^x}\right]$, then discuss the continuity at $x = 1$.

Sol$^n$. we know that

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x<1 \\ \infty & \text{if } x>1 \\ 1 & \text{if } x=1 \end{cases}$$

Therefore, If $x<1$, then.

$$f(x) = \lim_{n\to\infty}\left(\frac{x^n}{1+x^n e^x}\right) = \frac{0}{1+0} = 0 \quad \text{—— (i)}$$

If $x=1$, then

$$f(x) = \lim_{n\to\infty}\left(\frac{x^n}{1+x^n e^x}\right) = \frac{1}{1+e} \quad \text{—— (ii)}$$

If $x<1$, then.

$$f(x) = \lim_{n\to\infty}\left(\frac{x^n}{1+x^n e^x}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{x^{-n}+e^x}\right) = \frac{1}{0+e^x} = \frac{1}{e^x} \quad \text{—— (iii)}$$

$$\Rightarrow \quad f(x) = \begin{cases} 0 & \text{if } x<1 \\ \frac{1}{e^x} & \text{if } x>1 \\ \frac{1}{1+e} & \text{if } x=1 \end{cases}$$

$\Rightarrow$ LHL of $f(x)$ at $x=1$ $= f(1-) = 0$ —— (iv)

RHL of $f(x)$ at $x=1$ $= f(1+) = \frac{1}{e}$ —— (v)

V.O.$f^n$. at $x=1$ $= \frac{1}{1+e}$ —— (vi)

Here LHL $\neq$ RHL $\neq$ VOF$^n$.

$\Rightarrow$ Discontinous at $x=1$ & it is of first Kind of discontinuity.

Q/ Let the step function $f(x) = [x]$ is defined for $x\in\mathbb{R}$ s.t. $[x]$ = The greatest integer less than or equal to $x$. Then prove that $f(x)$ is not continous at any integral point.

Sol$^n$ Let $x = a$ be an integer.

then $\underset{x \to a^-}{Lt} f(x) = \underset{h \to 0}{Lt} f(a-h) = \underset{h \to 0}{Lt} [a-h]$

$= \underset{h \to 0}{Lt} (a-1) = a-1$ —— (i)

$\underset{x \to a^+}{Lt} f(x) = \underset{h \to 0}{Lt} f(a+h) = \underset{h \to 0}{Lt} [a+h]$

$= \underset{h \to 0}{Lt} (a) = a$ —— (ii)

V.O f$^n$ at $x = a$ $= [a] = a$ —— (iii)

Here LHL $\neq$ RHL $\Rightarrow$ f(x) is discontinuous at any integral point.